Sunspots and Predictable Asset Returns*

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Abstract

This paper uses a stylised asset-pricing model to show that sunspots may cause asset returns to be predictable, a widely documented feature of many speculative markets. This result parallels and extends previous works showing that sunspots render asset prices excessively volatile. *Journal of Economic Literature* Classification Numbers: D84, E44, G12

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1 Introduction

Some recent contributions (eg Azariadis and Chakraborty [1], Farmer [9]) have shown that sunspot fluctuations could generate the kind of excess volatility in asset prices that has been found empirically in some financial markets (see, for instance, Shiller [12] and Cambell and Shiller [3] for evidence on the US stock market). The purpose of the present note is to show that models open to the self-fulfilling prophecies of investors may also account for the predictability of asset returns, another widely documented feature of speculative markets (especially stock markets), both in univariate (eg [5, 7, 11]) and multivariate (eg [3, 4, 8]) contexts. Although the argument will be analysed within a simple – and deliberately stylised – asset-pricing model, this note aims to illustrate the fact that the predictability of asset returns is an immediate implication of sunspot fluctuations occurring near an indeterminate steady state, and can thus be expected to hold in a variety of models sharing similar properties.

The model is presented in section 2. Section 3 derives an approximate log-linear solution to the model, and section 4 analyses the time-series properties of asset returns implied by the model.

2 The Model

Overlapping generations models are frequently used in asset pricing, to represent economies where the horizon of investors is shorter than the life span of traded assets. The model presented here follows this approach by embedding a simple portfolio choice problem into Gale’s [10] ‘classical’ economy1. The economy is populated by overlapping generations of identical, two-period lived agents in large number, as well as by a central bank2. There is one good in the economy, used for consumption and investment purposes alike, whose nominal price at date $t$ is $P_t$. Units of the good can

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1In spite of Gale’s [10] early advocation, ‘classical’ economies have been less popular amongst macroeconomists than ‘Samuelson’ ones. Contributions analysing the occurrence of endogenous fluctuations in pure exchange, classical economies include Benhabib and Day [2], and Davila [6].

2Although other institutional devices are conceivable (eg. a regulated banking system with at par convertible monies, or a clearing house like that described by Gale), the one of a central bank is the simplest one in which the monetary institution spans more markets than private investors do.
be obtained either by investing in a safe capital intensive technology, or by buying long-lived assets yielding a stochastic dividend. Population grows at the constant rate $n$.

2.1 Technology

The capital-intensive technology is represented by a $C^2$ function $Y_t = f(K_t)$, where $K_t$ and $Y_t$ denote each individual’s investment and output, respectively. Production takes one period, so that $Y_t$, although known at date $t$, is only available at date $t + 1$. It is assumed, with no loss of generality, that capital fully depreciates after production.

Long-lived assets are in fixed supply, normalised to one. They yield a stochastic dividend $D_t$, known at date $t$ but paid at date $t + 1$, whose behaviour is described by a logarithmic random walk with drift:

$$\Delta \log D_t = g + \epsilon_t,$$

where $\{\epsilon_t\}_{t=0}^{\infty}$ is a white noise process with mean 0 and variance $\sigma^2_\epsilon > 0$.

Finally, the following assumptions are made:

i) $f'(K) > 0$, $f''(K) < 0$ and $f(0) = 0$;
ii) $f'(0) = \infty$ and $f'(\infty) = 0$;
iii) $-f''(K) K / f'(K) < 1 \forall K > 0$;
iv) $0 < g < n$

2.2 Investors’ behaviour

Investors enter the market at date $t$ with no endowment, and leave it at date $t + 1$. Agents are risk-neutral and derive no utility from first-period consumption\(^3\). They provide for late-life consumption, $C_{t+1}$, by buying goods (invested in the capital-intensive technology) and assets from investors who leave the market, using money they borrow from the bank. They leave the market at date $t + 1$ after having

\(^3\)This assumption is made for expositional ease. Similar results are obtained assuming risk-averse agents with time-separable utility over the entire life cycle, provided that the substitution effect sufficiently dominates the income effect.
sold some of their goods, together with their assets, in order to repay their loans.
Assuming, with no loss of generality, that the nominal interest rate charged by the
bank is zero, the budget and non-negativity constraints are:

\[ P_t (K_t + X_t Q_t) \leq M_t \]  \hspace{1cm} (2)
\[ P_{t+1} C_{t+1} + M_t \leq P_{t+1} (f (K_t) + X_t (Q_{t+1} + D_t)) \]  \hspace{1cm} (3)
\[ K_t, X_t, C_t \geq 0, \]  \hspace{1cm} (4)

where \( M_t \) denotes the individual demand for money, \( X_t \) the individual demand for
assets, and \( Q_t \) the price of assets in terms of goods, all at date \( t \). Investors maximise
terminal consumption conditionally on information available at date \( t \), and subject
to (2), (3) and (4). Thus, they solve the following portfolio choice problem:

\[ \{K_t, X_t\} \in \arg \max f(K_t) + X_t (E_t (Q_{t+1}) + D_t) - (K_t + X_t Q_t) E_t (P_t/P_{t+1}) \]  \hspace{1cm} (5)

As agents are risk neutral, they get no consumption premium for holding assets.
Hence, asset returns must satisfy the following no-arbitrage condition:

\[ E_t ((Q_{t+1} + D_t)/Q_t) = E_t (P_t/P_{t+1}) \]  \hspace{1cm} (6)

A price sequence that would not equalise expected returns \((E_t ((Q_{t+1} + D_t)/Q_t))\)
and expected costs \((E_t (P_t/P_{t+1}))\) on holding assets, would cause asset demand to
be zero or infinite, which would violate the equilibrium condition on the market for
assets. Using (6) to cancel assets from (5), the optimal choice of capital is such that:

\[ f' (K_t) = E_t (P_t/P_{t+1}) \]  \hspace{1cm} (7)

It is assumed that the first generation of ‘old’ investors initially hold the risky
assets as well as some units of the good. We follow Gale [10] and Benhabib and
Day [2] in assuming that they are initially in debt, and must repay \( M_0 \) units of
money to the bank. They then acquire this money by selling the assets (in quantity
1 and at nominal price \( P_0 Q_0 \)) and some of their goods (in quantity \( K_0 \) and nominal
price \( P_0 K_0 \)) to investors entering the market at date 0. Such a device initiates
the growth process described by Gale [10, Sections 3 and 5] for the case of pure
exchange economies. Note that \( K_0 \) and \( Q_0 \) are not initial conditions, but depend on
the expected course of the economy.
2.3 Market clearing and equilibrium asset prices

An equilibrium is defined as a sequence \( \{K_t, P_t, Q_t, X_t, M_t\}_{t=0}^{\infty} \) such that (6), (7), as well as the following market clearing conditions, hold at all dates:

\[
(1 + n) M_{t+1} = M_t \quad \text{(money market)} \tag{8}
\]

\[
N_t X_t = 1 \quad \text{(asset market)} \tag{9}
\]

\[
C_{t+1} + (1 + n) K_{t+1} = f(K_t) + D_t / N_t \quad \text{(good market)} \tag{10}
\]

Eq. (9) can hold only if (6) does. From equations (6) and (7), one can rewrite the no-arbitrage condition as:

\[
Q_t = (E_t(Q_{t+1}) + D_t) / f'(K_t) \tag{11}
\]

Finally, rational bubbles are ruled out by imposing the transversality condition:

\[
\lim_{n \to \infty} E_t \left( \prod_{i=0}^{n} \frac{Q_{t+i}}{f'(K_{t+i})} \right) = 0 \tag{12}
\]

Therefore, asset prices are always equal to their market fundamental, that is:

\[
Q_t = E_t \left( \sum_{i=0}^{\infty} \frac{D_{t+i}}{\prod_{j=0}^{i} f'(K_{t+j})} \right) \tag{13}
\]

3 Sunspots and the dividend/price ratio

3.1 Dynamics

Taking eq. (2) at two consecutive dates implies that, in equilibrium, the following relation holds:

\[
P_{t+1} / P_t = (1 + n) (K_{t+1} + X_{t+1} Q_{t+1}) / (K_t + X_t Q_t) \tag{14}
\]

Substituting (14) into (7), and using (9) and (11), one can summarise the dynamics of the economy by the following pair of difference equations:

\[
(1 + n) E_t(K_{t+1}) = K_t f'(K_t) + D_t / N_t \tag{15}
\]

\[
E_t(Q_{t+1} / Q_t) + \Lambda_t = f'(K_t) \tag{16}
\]

where \( \Lambda_t \) denotes the dividend/price ratio, \( D_t / Q_t \). The first equation is not au-
tonomous. However, assumption iv implies that:

$$\lim_{t \to \infty} D_t/N_t = 0$$  \hspace{1cm} (17)

Therefore, as $t \to \infty$, eq. (15) reduces to a familiar accumulation equation (eg the Solow growth model), whereas eq. (16) simply states that the gross expected return on holding assets (that is, capital gains plus the dividend yield) should be equal to marginal productivity. In the log-linearised model studied in the next subsection, we shall focus on the asymptotic behaviour of the system, using (17) to omit dividends from the approximated accumulation equation\(^4\).

3.2 An approximate log-linear solution

As in the pure exchange, classical economy studied by Gale, our production economy has both an autarkic and a golden-rule steady-state. Only the latter, where inter-generational trades of goods and assets take place, is of interest here. Eq. (15) implies that the asymptotic value of capital at the golden-rule steady state exists and is given by:

$$K^* = f'^{-1} (1 + n)$$

Eq. (13) implies that the corresponding steady-state price/dividend ratio is:

$$\sum_{i=0}^{\infty} \frac{E(D_{t+i}/D_t)}{\prod_{j=0}^{i} f'(K^*)} = \frac{1}{1 + n} \sum_{i=0}^{\infty} \left( \frac{1 + g}{1 + n} \right)^i = \frac{1}{n - g} ,$$

from which it follows that the golden-rule dividend/price ratio is $\Lambda^* = n - g$. Note that the constancy of the dividend/price ratio at the golden-rule steady-state implies that prices grow, on average, at the same rate as dividends $(E(Q_{t+1}/Q_t) = 1 + g)$.

Using lower-case letters to denote natural logarithms of the corresponding capital letters, the log-linearisation of eq. (15) and (16) around $(K^*, \Lambda^*)$ gives:

$$E_t (k_{t+1} - k^*) = (1 - \varepsilon) (k_t - k^*)$$ \hspace{1cm} (18)

$$\rho E_t (\Delta q_{t+1} - g) + (1 - \rho) (\lambda_t - \Lambda^*) = -\varepsilon (k_t - k^*) ,$$ \hspace{1cm} (19)

\(^4\)This amounts to pricing assets whose share in the whole portfolio of agents becomes infinitely small, as in Tirole [14].
where $\varepsilon \equiv -f''(K^*)K^*/f'(K^*)$, $\rho \equiv (1 + g)/(1 + n)$, and $\lambda_t = d_t - q_t$ is the log-dividend/price ratio. Using eq. (1), one can rearrange eq. (19) to obtain:

$$-\rho E_t (\lambda_{t+1} - \lambda^*) + (\lambda_t - \lambda^*) = -\varepsilon (k_t - k^*)$$  \hspace{1cm} (20)

Equations (18) and (20) yield the following dynamic system:

$$\begin{bmatrix} k_t - k^* \\ \lambda_t - \lambda^* \end{bmatrix} = \begin{bmatrix} 1/ (1 - \varepsilon) & 0 \\ -\varepsilon/ (1 - \varepsilon) & \rho \end{bmatrix} \begin{bmatrix} E_t (k_{t+1} - k^*) \\ E_t (\lambda_{t+1} - \lambda^*) \end{bmatrix}$$

Or, more concisely:

$$V_t = ME_t (V_{t+1})$$  \hspace{1cm} (21)

The stability properties of eq. (21) depend on the comparison between the number of predetermined variables in $V_t$, and the number of roots in $M$ that lie outside the unit circle. From assumptions iii and iv, $M$ has one forward-stable root, $\rho \in (0, 1)$, and one backward-stable root, $1/ (1 - \varepsilon) \in (1, \infty)$. Since none of the variables in the vector $V_t$ is predetermined, the steady-state $(k^*, \lambda^*)$ is indeterminate. It is shown in the appendix that the general solution to eq. (21) yields the following AR(1) process for the dividend/price ratio:

$$\lambda_t - \lambda^* = (1 - (1 - \varepsilon) L)^{-1} \eta_t$$  \hspace{1cm} (22)

where $L$ is the lag operator, and where $\{\eta_t\}_{t=0}^\infty$ is a white noise process with mean 0, variance $\sigma_\eta^2$ and small bounded support, that shall be interpreted as a sequence of expectational disturbances, or ‘sunspots’. Figure 1 offers a heuristic, phase-diagrammatic representation of the solution dynamics. Sunspots trigger cumulative, local jumps of $(k_t, \lambda_t)$ along the stable branch of the saddle, which temporarily drive the equilibrium away from $(k^*, \lambda^*)$. The stable branch slopes downwards, as a high level of capital is associated with a low equilibrium interest rate, high asset prices, and hence a low dividend/price ratio.

\textbf{INSERT FIGURE 1}

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5One may notice that the left hand-side of eq.(19) is similar to Campbell and Shiller’s [3] log-linear approximation for expected stock returns (ie where price growth, rather than the price level, is approximated).
4 Time-series implications

4.1 From the dividend/price ratio to prices and returns

Eq. (22) can be rearranged to obtain the following moving-average representation for the level of asset prices:

$$q_t = -\lambda^* + d_t - \sum_{j=0}^{\infty} (1 - \varepsilon)^j \eta_{t-j}$$

Eq. (23) has the following interpretation – asset prices and dividends are cointegrated; dividends drive the stochastic trend in asset prices ($-\lambda^* + d_t$), whereas sunspots entirely account for their cyclical variations ($-\eta_t / (1 - (1 - \varepsilon) L)$), i.e. their transitory departures from the common trend they share with dividends.

Differencing eq. (23) yields the following process for ex post, continuously compounded returns:

$$\Delta q_t = g + \varepsilon_t - \eta_t + \varepsilon \sum_{j=0}^{\infty} (1 - \varepsilon)^j \eta_{t-j-1}$$

The next two subsections analyse the implications of the sunspot-driven cycle in asset values, for the predictability of asset returns.

4.2 Univariate returns predictability

Eq. (23) implies that sunspots have a persistent effect on asset prices, which has important implications for the serial correlation of asset returns. Assuming, to simplify the discussion of the results, that sunspots and dividend shocks are uncorrelated$^6$, the returns’ auto-correlation function generated by eq. (24) is:

$$\varphi(n) = \frac{\text{cov}(\Delta q_t, \Delta q_{t-n})}{\text{var}(\Delta q_t)} \quad (n = 1, 2, \ldots)$$

$$= -\frac{\varepsilon (1 - \varepsilon)^n \sigma^2}{(2 - \varepsilon) \sigma^2 + 2 \sigma^2} \leq 0$$

$^6$A non-zero covariance between sunspot and dividend innovations, possibly interpreted as ‘over-reaction’ ($\text{cov}(\varepsilon_t, \eta_t) < 0$) or ‘under-reaction’ ($\text{cov}(\varepsilon_t, \eta_t) > 0$) to dividend news, may change the auto-correlation pattern of returns (namely, $\text{cov}(\varepsilon_t, \eta_t) > \sigma^2 / (2 - \varepsilon)$ makes ex post returns positively auto-correlated). Setting $\text{cov}(\varepsilon_t, \eta_t) = 0$ implies that eq. (23) has an Unobserved Components representation similar to that frequently used in applied studies (eg [7, 11, 13]).
Sunspots cause returns to be negatively serially correlated, and thus predictable on the basis of past returns (especially when $n$ is ‘small’). Similar auto-correlation patterns have been the focus of many empirical studies, including those of Fama and French [7], Poterba and Summers [11] and Cutler et al. [5]. Intuitively, the negative serial correlation in asset returns comes from the fact that sunspots trigger trend-reverting fluctuations in asset-prices (see eq. (23)). This implies that the positive (negative) returns accompanying a transitory increase (decrease) in asset prices, tend to be followed by negative (positive) returns as prices revert towards their stochastic trend. Note, however, that the $\varphi(n)$s are small if $\sigma^2_\eta$ is small relative to $\sigma^2_\epsilon$, in which case sunspots might only induce little univariate predictability. Nevertheless, they would still induce strong multivariate predictability.

4.3 Multivariate returns predictability

A property of the above model is that dividends provide relevant incremental information for the forecasting of future returns, even though dividends are not themselves predictable at all (see eq. (1)). To understand why this is the case, rearrange eq. (24) to obtain:

$$\Delta q_t = g + \varepsilon (\lambda_{t-1} - \lambda^*) + \epsilon_t - \eta_t$$  \hspace{2cm} (25)

Eq. (25) has an error-correcting structure; any departure of the dividend/price ratio, $\lambda_t$, from its long-run average, $\lambda^*$, triggers a change in asset prices, $\Delta q_{t+1}$, that contributes to restore the equilibrium value of the ratio. This error-correction mechanism implies that asset returns can be predicted on the basis of past dividend/price ratios, as has been documented in a number of empirical studies (see, among others, Campbell and Shiller [3], Fama and French [8], and Cochrane [4]). To illustrate this point more concretely, suppose that we use (1) and (23) to generate (large) artificial series of dividends and prices, and then go on running the following first-order, error-correction model on those series:

$$\Delta q_t = \alpha + \beta \Delta q_{t-1} + \gamma \lambda_{t-1} + \zeta_t$$  \hspace{2cm} (26)

Comparing the data generation process (eq. (25)) and the corresponding regression (eq. (26)) indicates that the latter is correctly specified. Hence, OLS estimators
consistently pick the parameters of the underlying economic model and, if (and only if) $\sigma_\eta^2 \neq 0$, yield:

$$\hat{\beta} = 0, \quad \hat{\gamma} = \varepsilon$$

Running a regression similar to eq. (26) on data from the New-York Stock Exchange, Cochrane [4] found that the dividend/price ratio was the only significant predictor of stock returns once included in the regression, as is implied by (27). Moreover, multivariate predictability is stronger than univariate predictability, as the former does not depend on the relative size of dividend and sunspot innovations (ie as long as $\sigma_\eta^2 > 0$). That returns are weakly predictable on the basis of past returns, but strongly predictable on the basis of past dividend/price ratios, is a common result in the empirical literature on stock market volatility (see, for instance, [3, 4, 8, 11, 13]).
5 Appendix

First, (21) is rewritten as $V_t = S \Pi S^{-1} E_t (V_{t+1})$, where $\Pi$ is a matrix of eigenvalues, and $S$ is the corresponding matrix of eigenvectors (normalised, say, in such a way that their second coordinates are 1). Then, premultiplying the resulting equation by $S^{-1}$, and denoting $Z_t = \begin{bmatrix} z'_t & z_t \end{bmatrix}' \equiv S^{-1} V_t$ as the vector of transformed variables, the dynamics of the latter can be written as $Z_t = \Pi E_t (Z_{t+1})$, and that of its elements as $z_t = (1 - \varepsilon)^{-1} E_t (z_{t+1})$ and $z'_t = \rho E_t (z'_{t+1})$. Since $(1 - \varepsilon)^{-1} > 1$, any process of the form:

$$z_t = (1 - \varepsilon) z_{t-1} + \eta_t,$$

where $\{\eta_t\}_{t=0}^\infty$ is a white noise process with mean 0, solves $z_t = (1 - \varepsilon)^{-1} E_t (z_{t+1})$. On the other hand, since $\rho \in (0, 1)$, the only bounded sequence satisfying $z'_t = \rho E_t (z'_{t+1})$ is:

$$z'_t = 0$$

for all $t$. Let $Z_t = JZ_{t-1} + U \eta_t$ be the VAR representation of eq. (28) and (29). Premultiplying it by $S$ gives $V_t = SJS^{-1} V_{t-1} + SU \eta_t$, that is:

$$\begin{bmatrix} k_t - k^* \\ \lambda_t - \lambda^* \end{bmatrix} = \begin{bmatrix} 1 - \varepsilon & 0 \\ (1 - \varepsilon) \varepsilon / (\rho (1 - \varepsilon) - 1) & 0 \end{bmatrix} \begin{bmatrix} k_{t-1} - k^* \\ \lambda_{t-1} - \lambda^* \end{bmatrix}$$

$$+ \begin{bmatrix} (\rho (1 - \varepsilon) - 1) / \varepsilon \\ 1 \end{bmatrix} \eta_t$$

Noting that $k_t - k^* = ((\rho (1 - \varepsilon) - 1) / \varepsilon) (\lambda_t - \lambda^*)$, the second equation yields (22).

References


Figure 1. Phase diagram