Incomplete markets, liquidation risk, and the term structure of interest rates

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Abstract

We study the term structure of real interest rates within a general equilibrium model with incomplete markets and borrowing constraints. Agents are subject to both aggregate and idiosyncratic income shocks, which latter may force them into early portfolio liquidation in a bad aggregate state. We derive a closed-form equilibrium with limited agent heterogeneity (despite market incompleteness), which allows us to derive analytical expressions for bond prices and returns at any maturity. The attractiveness of bonds as liquidity makes the aggregate bond demand downward-sloping. One consequence is that a greater bond supply raises both the level and the slope of the yield curve.

Keywords: incomplete markets, yield curve, credit constraints.

JEL codes: E21, E43, G12.

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1 Introduction

It has frequently been argued that incomplete market models overcome some of the difficulties encountered in the complete market framework. They notably help to explain important features of asset prices, such as the level of the risk-free interest rate (Huggett (1993)), the size of the equity premium (Constantinides and Duffie (1996)), or the existence of money (Bewley (1983); Scheinkman and Weiss (1986)). These contributions are based on the central idea that market incompleteness introduces a particular motive to demand (and actually trade) available assets, since these are used to smooth out the idiosyncratic income fluctuations that agents face.\(^1\)

The term structure of interest rates is one of the few direct sources of information on agents’ expectations about future economic activity. Despite the potential insights to be gained, the term structure has only been the subject of surprisingly little research in the context of uninsurable idiosyncratic shocks. One potential explanation is the inherent complexity of infinite-horizon, incomplete market models with a large number of assets. On the one hand, market incompleteness usually implies that agents’ wealth and optimal decisions depend on the whole history of the idiosyncratic income shocks, so that infinitely many types of agents (with their associated Euler equations) co-exist in the economy. This usually precludes the derivation of analytical expressions and general conclusions regarding asset prices, except in very special cases.\(^2\) On the other hand, computational techniques, when applied to economies with aggregate shocks, can only handle a small number of assets, typically one or two (e.g., Krusell and Smith (1997) and Heathcote (2005)).

In this paper we analyze the term structure of real interest rates in an incomplete market, general equilibrium model that features a single good, infinitely-lived agents, and arbitrarily many bond maturities. We construct an equilibrium in which the entire term structure of interest rates, including the yields and returns of arbitrarily long bonds, can be characterized analytically. One property of this equilibrium is that asset liquidation is full, in the sense

\(^1\)Empirically (e.g., Zeldes (1989)), there is considerable evidence that households face more idiosyncratic risks than what it is implied by the complete market assumption.

\(^2\)For example, the no-trade equilibrium of Constantinides and Duffie (1996) and the deterministic liquidity constrained model of Kehoe and Levine (2001)
that agents facing a drop in their current earnings entirely liquidate their bond portfolio.

In our model, agents face uninsurable idiosyncratic shocks and a borrowing constraint. The economy is also hit by aggregate productivity shocks. The idiosyncratic income risk is specified as a random switch of agents’ status between ‘employment’ and ‘unemployment’. Unemployed agents earn a (low) home production income, while employed agents’ wages depend on the productivity shock. Agents’ attempts to smooth out their labor income fluctuations have two consequences: (i) When employed, agents accumulate bonds for self-insurance purposes; and (ii) when they become unemployed, they liquidate their bonds. The government issues zero-coupon bonds of various maturities, whose payoff is financed by non-distorting taxes.

We investigate the departure of our model from complete market economies by analyzing the effect on the yield curve of a change in the net supply of government bonds. This change can be regarded as exogenous variations in the amount of liquidity, defined as the assets available to self-insure against idiosyncratic shocks. While this change would not alter the yield curve under complete markets, aggregate bond demand is shown to be downward-sloping in our model: Increasing the supply of bonds of any maturity lowers the price of all bonds, i.e. it raises the entire yield curve. This is easily understood from the liquidity role played by bonds in our economy. In the presence of both idiosyncratic income risk and trade restrictions (i.e. debt limits), high-income agents hold bonds of any maturity for precautionary purposes. In this context, more liquidity lowers its desirability and equilibrium prices. Since bonds of various maturities are partly substitutes for each other, raising the supply of one particular type of bond must lower the price of all bonds.

Our second result is that a larger bond supply steepens the yield curve by affecting relative prices, i.e., the risk premia associated with bonds of different maturities. In our model, risk premia differ across bonds, because agents may be forced to liquidate assets before their maturity, when their selling price is low due to a bad realization of aggregate uncertainty. Since the risk of early liquidation increases with bond maturity, long bonds must command a greater premium in equilibrium than comparatively shorter bonds. Then,
raising the supply of bonds of any maturity steepens the yield curve, because it increases the overall variability of portfolios’ liquidation values. Thus, both the level and the slope of the yield curve go up with the supply of bonds.\footnote{See Fleming (2002) for empirical evidence on the impact of net bond supply on the yield curve consistent with these results. See also Jegadeesh (2002) for a more general discussion of this evidence.}

Finally, we explore the welfare implications of our model. More specifically, we establish that while increasing the quantity of bonds of any maturity unambiguously raises \textit{ex ante} welfare (i.e. from the point of view of date 0, before agents know their type), it may either increase or decrease the \textit{ex post} welfare of agents, depending on their employment status and subjective rate of time preference. In a nutshell, agents immediately suffer from higher taxes, but will benefit, from the next period onwards, from the implied improvements in self-insurance possibilities. For the second effect to dominate, agents must be sufficiently patient.

Following the seminal work of Vasicek (1977) and Cox, Ingersoll, and Ross (1985), several authors have documented the inability of baseline general equilibrium models with complete markets to account for the main empirical features of the yield curve. For example, complete market models typically fail to reproduce the slope of the yield curve (e.g., Donaldson, Johnsen, and Mehra (1990)), the volatility of interest rates (e.g., Buraschi and Jiltsov (2007)), as well as the rejection of the ‘expectation hypothesis’ (see Backus, Gregory, and Zin (1989)).\footnote{The expectation hypothesis states that forward rates are unbiased predictors of future spot rates. See Campbell and Schiller (1991) for the evidence on the empirical failure of the expectation hypothesis.}

The idea that incomplete markets could contribute to the resolution of these puzzles has first been explored in finite horizon economies. For example, Heaton and Lucas (1992) and Holmstrom and Tirole (2001) use three-period models to analyze the effects on the yield curve of the interaction between idiosyncratic and aggregate risks. These models leave open the impact of this interaction at longer horizons.

Seppälä (2004) computes the equilibrium yield curve in an economy à la Alvarez and Jermann (2000), where two agents interact in an endowment economy with a complete set of contingent claims and an endogenous participation constraint. Following the definition of Kehoe and Levine (2001), his economy features ‘debt constrained’ markets, while ours has
‘liquidity constrained’ markets. By construction, the two-agent economy of Seppälä does not distinguish between aggregate and idiosyncratic uncertainties, whose interaction is the focus of our study.

Finally, an alternative approach has been to assume the absence of arbitrage and to directly consider a pricing kernel to price bonds of various maturities (Ho and Lee (1986) among others). Some recent papers following this methodology introduce macroeconomic factors as determinants of the pricing kernel (e.g., Ang and Piazzesi (2003)). Deriving the pricing structure from utility maximization, our paper identifies the channels through which macroeconomic conditions affect the term structure of the interest rate. In addition, it may help to identify new macroeconomic factors, such as the incidence of fiscal policy.\footnote{Dai and Philippon (2006) considers the level of public debt as a macroeconomic factor.}

The remainder of the paper is organized as follows. We introduce our framework in Section 2. Section 3 describes the equilibrium with full asset liquidation, while Section 4 establishes the conditions for its existence. Section 5 studies the impact of changes in bond supply on the shape of the yield curve. Section 6 derives the welfare properties of the model. Section 7 concludes and discusses further results that can be derived within this framework.

2 The economy

2.1 Individual and aggregate states

In every period, each agent can be in either of two states, ‘employed’ or ‘unemployed’. Let $e_i^t$ denote the status of agent $i$ at date $t$, where $e_i^t = 1$ if the agent is employed and $e_i^t = 0$ if the agent is unemployed. Each agent’s employment status in the labor market evolves independently according to a first-order Markov chain with the following transition matrix:

$$\Pi = \begin{bmatrix} \alpha & 1 - \alpha \\ 1 - \rho & \rho \end{bmatrix}, \quad (\alpha, \rho) \in (0, 1)^2,$$
where $\alpha$ is the probability that an employed agent stays employed in the next period and $\rho$ is the probability that an unemployed agent stays unemployed in the next period.

The initial probability distribution is represented by a row vector $\omega_0 = \begin{bmatrix} \omega^e_0 & \omega^u_0 \end{bmatrix}$, i.e. $\omega^e_0$ (respectively $\omega^u_0$) is the probability at date 0 that agent $i$ is employed (unemployed).

Given this simple Markovian structure, the probability distribution at date $t$, which is $\omega_0 \Pi^t$, converges for $t \to \infty$ to the invariant distribution $\omega = \begin{bmatrix} \omega^e & \omega^u \end{bmatrix}$, where:

$$
\omega^e = (1 - \rho) (2 - \rho - \alpha)^{-1}, \quad \omega^u = (1 - \alpha) (2 - \rho - \alpha)^{-1}.
$$

(1)

To simplify the exposition, we assume that $\omega_0 = \omega$ (i.e. the initial proportions of employed and unemployed agents are at the invariant distribution level).

The history of individual shocks up to date $t$ is denoted by $e^{i,t}$, where $e^{i,t} = \{e^i_0, \ldots, e^i_t\} \in \{0,1\}^t = E^t$. $E^t$ is the set of all possible individual histories up to date $t$, and $\mu^i_t : E^t \to [0,1]$, $t = 0, 1, \ldots$ denotes the probability measure of individual histories, consistent with the transition matrix $\Pi$ and the initial probability distribution $\omega$. For example, $\mu^i_t(e^{i,t})$ is the probability that agent $i$ experiences the history $e^{i,t}$ at date $t$. We use the notation $e^{i,t+1} \succeq e^{i,t}$ to indicate that $e^{i,t+1}$ is a possible continuation of $e^{i,t}$.

### 2.2 Aggregate states

The economy faces an aggregate (technology) shock, whose value at date $t$ is $h_t$. The value of this shock can be either high ($h_t = h$) or low ($h_t = l$). Let $h^t = \{h_0, \ldots, h_t\}$ denote the history of aggregate shocks from date 0 to date $t$, and $H^t$ be the set of all such possible histories. The aggregate shock evolves according to a first-order Markov chain with transition matrix $T$:

$$
T = \begin{bmatrix}
\pi^h & 1 - \pi^h \\
1 - \pi^l & \pi^l
\end{bmatrix}
$$

Moreover, we make the following assumption:

**Assumption A** $\pi^h + \pi^l > 1$.  

6
Assumption A is a statement about the persistence of aggregate shocks, implying that the economy does not fluctuate too quickly between the two aggregate states. While not necessary for the derivation of most of our results, it allows us to avoid discussing some uninteresting cases arising from rapidly alternating states.

The invariant distribution associated with the transition matrix $T$ is denoted $\Phi = [\Phi_l \, \Phi_h]$. We assume that the initial probability distribution across both aggregate states is $\Phi$. We denote by $\nu_t$ the probability measure over histories up to date $t$, consistent with the transition matrix $T$, and the initial distribution $\Phi$: $\nu_t : H^t \rightarrow [0,1], t = 0, 1, \ldots$ Finally, we denote by $\nu_t(h^t)$ the probability that history $h^t$ occurs, and $h^{t+1} \succeq h^t$ indicates that $h^{t+1}$ is a possible continuation of $h^t$.

2.3 Assets and market structure

The only assets that agents may buy and sell are riskless zero-coupon bonds. These pay off one unit of goods at maturity, and are issued by the government. The maturities of these bonds vary from 1 to $n \geq 1$, where $n$ may be arbitrarily large. A bond of maturity $k > 1$ at date $t$ becomes a bond of maturity $k - 1$ at date $t + 1$, and eventually yields 1 at date $t + k$. The price of this bond at date $t$ is denoted $p_{t,k}(h^t)$. To simplify notations, we define the price of a bond of maturity 0 as its payoff: $p_{t,0}(h^t) = 1$.

Our assumption that the only tradable assets are government bonds means that private agents cannot themselves issue any securities. This has three significant implications. First, there is no asset providing a payoff contingent on agents’ idiosyncratic employment status: Unemployment risk is thus entirely uninsurable. Second, agents face an exogenous debt limit preventing them from issuing debt securities in both aggregate states. Third, there is no security offering a payoff contingent on the aggregate state of the economy. These properties are the heart of a vast literature on liquidity constrained economies since the seminal work of Bewley (1980).

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6 This assumption appears empirically relevant: Using US quarterly data on GNP, Hamilton (1994), chap 22, finds that $\pi^h + \pi^l = 1.65$.

7 See Kehoe and Levine (2001) for references on related papers.
Bond payoffs are financed by both the issuing of new government bond and taxes. At each date \( t \), a given net quantity \( A_{t,k} \) of zero coupon bonds paying one unit of the good at period \( t + k \) is issued by the government at the price of \( p_{t,k}(h^t) \). The government repays bonds arriving at maturity at date \( t \).

In order to minimize tax distortions, we assume that the government uses lump-sum transfers contingent on employment status. The unemployed pay no tax, while all employed agents pay the same amount \( \tau^e \). Given that there is a proportion \( \omega^e \) of the latter, the government budget constraint is given by:

\[
\sum_{k=1}^{n} p_{t,k} A_{t,k} + \omega^e \tau^e_t = \sum_{k=1}^{n} A_{t-k,k}
\]

The aggregate supply of securities with a given maturity is composed of newly-issued bonds of that maturity plus longer bonds issued earlier and coming closer to maturity. At date \( t \), a total quantity \( B_{t,k} \) of bonds reaching maturity at date \( t + k \) is available in the market, where:

\[
B_{t,k} \equiv \sum_{j=0}^{n-k} A_{t-j,k+j}
\]

For the sake of conciseness, we focus on the case where the quantity of bonds of a given maturity is constant (i.e. \( B_{t,k} = B_k, \forall t \geq 0 \)), but the model can easily accommodate stochastic changes in bond supply. Constant bond quantities per maturity is equivalent to constant issuances, i.e. \( A_{t,k} = A_k, \forall t \geq 0 \).

### 2.4 Firms

There are a large number of perfectly competitive firms, which produce a single final good using only labor with constant returns-to-scale technology. Labor productivity, \( z_t \), depends on the aggregate state of the economy. Productivity levels in states \( h \) and \( l \) are \( z^h \) and \( z^l \) respectively, where \( z^h \geq 1 \geq z^l > 0 \). Firm profit maximization under perfect competition

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8Introducing either proportional taxes on labor income or lump sum taxes on the unemployed would generate additional distortions and/or redistributive effects, whose impacts are beyond the scope of this paper.
implies that the real wage $w_t$ equals the marginal product of labor, i.e. $w_t = z_t$.

### 2.5 Agents

There is a continuum of agents of mass 1. Each agent $i$ has preferences over consumption and labor. We follow Scheinkman and Weiss (1986) in assuming that intertemporal preferences are time-separable and that agents’ instantaneous utility is $u(c) - l$, where $c$ is consumption, $l$ is labor supply, and $u$ is a $C^2$ function satisfying $u'(\cdot) > 0$, and $u''(\cdot) < 0$. The marginal disutility of labor is constant and normalized to 1. All agents discount their instantaneous utility using the same factor $\beta \in (0, 1)$.

In period $t$, each agent $i$ consumes an amount $c^i_t$, supplies labor $l^i_t$, and demands the quantity $b^i_{t,k}$ of bonds with maturity $k$. The agent also pays a lump-sum tax $\tau^i(e^i_t)$, which is contingent on their employment status. Employed agents choose their labor supply and earn a hourly wage equal to $w_t$. Unemployed agents earn no labor income, but a constant amount of home production $\delta \geq 0$. Their labor supply is by definition equal to 0. We denote by $b^i_{t-1,k}$ the quantity of $k$-period bonds that agent $i$ holds at the beginning of period 0. Particular assumptions about initial bond holdings will be made in Section 4 in order to simplify the transitional dynamics of the model.

Agent $i$’s problem consists in choosing the following sequences of functions:

\[
\begin{align*}
  c^i_t & : H^t \times E^t \rightarrow R_+ \\
  l^i_t & : H^t \times E^t \rightarrow R_+ \\
  b^i_{t,k} & : H^t \times E^t \rightarrow R_+ \quad k = 1, \ldots, n
\end{align*}
\]

These choices maximize the agent’s intertemporal expected utility:

\[
\sum_{t=0}^{\infty} \beta^t \sum_{h^t \in H^t} \nu_t(h^t) \sum_{e^i,t \in E^t \mu^i_t(e^i,t}) (u(c^i_t(h^t, e^i,t)) - l^i_t(h^t, e^i,t))
\]
subject to the following constraints:

\[ c_i^t(h^t, e^{i,t}) + \tau_t(e_i^t) + \sum_{k=1}^{n} p_{t,k} (h^t) b_{t,k}^i (h^t, e^{i,t}) = \sum_{k=1}^{n} p_{t,k-1} (h^t) b_{t-1,k}^i (h^{t-1}, e^{i,t-1}) \]

\[ + e_i^t z_t^i (h^t, e^{i,t}) + (1 - e_i^t) \delta \] (2)

\[ c_i^t(h^t, e^{i,t}), \quad l_i^t(h^t, e^{i,t}) \geq 0 \] (3)

\[ b_{t,k}^i(h^t, e^{i,t}) \geq 0 \] (4)

\[ \lim_{t \to \infty} \beta^t u'(c_i(h^t, e^{i,t})) b_{t,k}^i(h^t, e^{i,t}) = 0 \quad \text{for} \quad k = 1, \ldots, n \] (5)

The budget constraint (2) sets the agent’s receipts and expenditures equal at date \( t \). The total receipts of agent \( i \) at date \( t \) are the sum of the sale value of the bond portfolio on the one hand, and on the other hand labor income if \( e_i^t = 1 \) or home production income if \( e_i^t = 0 \). The agent’s current income is spent on consumption goods and bonds of various maturities, as well as on paying for lump sum taxes. Constraint (4) reflects that agents cannot issue any debt securities. Finally, (5) is the set of transversality conditions, which always hold along the equilibrium we will be considering.

We also make the following assumption:

**Assumption B** \( 1/z^i < u'(\delta) \)

This assumption implies that, in equilibrium, the marginal utility of the consumption enjoyed by an unemployed agent is always higher than that enjoyed by an employed agent (i.e. the unemployed are always worse off than the employed in both aggregate states). In other words, the utility gain from one unit of labor paid at the lowest wage provides an upper bound for the income of the jobless \( \delta \).
The Lagrangian function associated with agent $i$’s problem is as follows:

$$L = \sum_{t=0}^{\infty} \beta^t \sum_{h^t \in H^t} \nu_t (h^t) \sum_{e^{i,t} \in E^t} \mu_t (e^{i,t}) \times \left[ u \left( c^i_t (h^t, e^{i,t}) \right) - l^i_t (h^t, e^{i,t}) + \sum_{k=1}^{n} \varphi^i_{t,k} (h^t, e^{i,t}) b^i_{t,k} (h^t, e^{i,t}) ight]$$

$$+ \eta^i_t (h^t, e^{i,t}) \left( e^i_t z_t l^i_t (h^t, e^{i,t}) + (1 - e^i_t) \delta - \tau_t (e^i_t) + \sum_{k=1}^{n} p_{t,k-1} (h^t) b^i_{t-1,k} (h^{t-1}, e^{i,t-1}) + c^i_t (h^t, e^{i,t}) \right)$$

The Lagrange multipliers $\eta^i_t$ and $\varphi^i_{t,k}$ are positive functions defined over $H^t \times E^t$, and are associated with the budget constraint (2) and the borrowing constraint (4), respectively. (We check below that the non-negativity constraints (3) for $c^i_t$ and $l^i_t$ are always satisfied in the equilibrium we consider). From the Kuhn-Tucker theorem, the optimality conditions are, for $t = 0, 1, \ldots$ and for all $(h^t, e^{i,t}) \in H^t \times E^t$:

$$u' \left( c^i_t (h^t, e^{i,t}) \right) = \eta^i_t (h^t, e^{i,t})$$

$$\begin{cases} 
\eta^i_t (h^t, e^{i,t}) & = 1/z_t \quad \text{if } e^i_t = 1 \\
l^i_t (h^t, e^{i,t}) & = 0 \quad \text{if } e^i_t = 0 
\end{cases}$$

$$\eta^i_t (h^t, e^{i,t}) p_{t,k} (h^t) = \beta \sum_{h^{t+1} \geq h^t} \nu_{t+1} (h^{t+1}) \sum_{e^{i,t+1} \geq e^{i,t}} \mu_{t+1} (e^{i,t+1}) \eta^i_{t+1} (h^{t+1}, e^{i,t+1}) p_{t+1,k-1} (h^{t+1})$$

$$+ \varphi^i_{t,k} (h^t, e^{i,t}) \quad \text{for } k = 1, \ldots, n$$

$$\begin{cases} 
\varphi^i_{t,k} (h^t, e^{i,t}) > 0 \quad \text{and } b^i_{t,k} (h^t, e^{i,t}) = 0 \quad \text{for } k = 1, \ldots, n \\
or \varphi^i_{t,k} (h^t, e^{i,t}) = 0 \quad \text{or } b^i_{t,k} (h^t, e^{i,t}) > 0 \quad \text{for } k = 1, \ldots, n 
\end{cases}$$

Equation (6) defines agent $i$’s marginal utility of consumption, while equation (7) describes optimal labor supply. When the agent works ($e^i_t = 1$), the marginal gain in consumption and the marginal disutility of labor are set equal (i.e. $1/u' (c^i_t (h^t, e^{i,t})) = 1/z_t$), while labor supply is zero when the agent is unemployed ($e^i_t = 0$). Equation (8) is the intertemporal optimality condition, which can also be written more compactly as:

$$u' \left( c^i_t \right) p_{t,k} = \beta E_t \left[ u' \left( c^i_{t+1} \right) p_{t+1,k-1} \right] + \varphi^i_{t,k}$$
where $E_t[.]$ is the expectation over both aggregate and idiosyncratic states, conditional on the information available at date $t$ (here $h^t$ and $e^{i,t}$). The Euler equation (10) sets equal the marginal cost of acquiring one unit of bonds of each maturity today with the marginal gain associated with its payoff tomorrow. When the shadow cost of the borrowing constraint is positive, meaning that the constraint is actually binding ($\varphi^i_{t,k}(h^t, e^{i,t}) > 0$), then agent $i$ would like to increase his expected utility by issuing $k$-period bonds (but he is prevented from doing so by assumption). Equation (9) finally summarizes the relationship between the shadow-cost $\varphi^i_{t,k}$ and the bindingness of the borrowing constraint.

2.6 Market clearing

The bond markets clear at date $t$ when the bond supply for each maturity equals the bond demand for the same maturity, i.e.:

$$\int_0^1 b^i_{t,k}(h^t, e^{i,t}) \, di = B_k, \quad \forall k = 1, \ldots, n.$$  \hspace{1cm} (11)

By Walras’ Law, the good market clears when all bond markets clear.

3 Equilibrium with full asset liquidation

One implication of our particular market structure is that the available assets play the role of a buffer stock, since they allow agents to partially offset the lack of full credit and insurance markets. However, many models of this class imply a smooth portfolio rebalancing in equilibrium: High-income agents gradually build up their asset wealth, while low-income agents decumulate assets at a sufficiently slow pace so as to never actually hit the borrowing constraint (e.g. Scheinkman and Weiss (1986), or Aiyagari (1994)). Since we focus on the implications of liquidation risk for bond pricing, we construct our equilibrium in such a way that agents do indeed liquidate assets when a negative idiosyncratic income shock hits (i.e. $b^i_{t,k}(h^t, e^{i,t}) = 0$ for $k = 1, \ldots, n$ if $e^i_t = 0$). All low-income agents therefore face a binding credit constraint (i.e., their labor income fall is not entirely offset by the portfolio liquidation
value). Incidentally, this full liquidation of bond holdings drastically reduces the number of agent types in the economy, thereby allowing us to study bond pricing analytically for an arbitrarily large number of maturities.

Our equilibrium is obtained by construction: We first conjecture, and then derive a sufficient condition for, the existence of an equilibrium along which employed agents are never borrowing-constrained (i.e. they are willing to save and thus hold a positive quantity of bonds at the end of the current period), while the unemployed always hit the borrowing constraint (i.e. they would like to borrow, rather than hold positive bond holdings, at the end of the current period). From (9), this joint conjecture can formally be written as:

\[
\begin{cases}
    \text{If } e_i^t = 1 \text{ then } \varphi_{i,k}^{t} (h^t, e_i^t) = 0 \\
    \text{If } e_i^t = 0 \text{ then } \varphi_{i,k}^{t} (h^t, e_i^t) > 0
\end{cases}
\quad \text{for all } k = 1, \ldots, n \tag{12}
\]

3.1 Equilibrium consumption levels

We first consider the consumption of an unemployed agent in period \( t \). If the agent was employed in the previous period, then from the budget constraint (2) and the conjecture (12), the agent earns the home production \( \delta \) and the liquidation value of the portfolio. Since he is borrowing constrained, the agent consumes his entire income. His consumption can therefore be written as:

\[
c_i^t = \sum_{k=1}^{n} p_{t,k-1} b_{t-1,k} + \delta \quad (> 0)
\quad \tag{13}
\]

The consumption of unemployed agents, who were already unemployed in the previous period, is identical across agents. They earn the home production \( \delta \) and cannot raise consumption through borrowing. Their current consumption, denoted \( c_{t}^{uu} \), is thus:

\[
c_{t}^{uu} = \delta \quad (> 0)
\quad \tag{14}
\]

We now turn to employed agents. From their intratemporal optimality conditions (6) and (7), the consumption level, denoted \( c_{t}^{e} \), is identical across employed agents. Employed agents set the marginal utility of consumption equal to the marginal disutility of labor.
The constant disutility of labor pins down consumption, which depends only on preference parameters and the aggregate shock:

$$c^e = u'^{-1}(1/z_t) \ (> 0) \quad (15)$$

If the agent is employed in the next period, which occurs with the probability $\alpha$, then $\eta_{t+1} = 1$ (see Eq. (7)). If the agent moves into unemployment the next period, then from (6) $\eta_{t+1} = u'(c_{t+1}^i)$, where by construction $c_{t+1}^i$ is given by the equation (13). From the first-order conditions (6), (7), and (10) of the agent’s problem and the equation (12), the Euler equations for employed agents express as $(k = 1, \ldots, n)$:

$$\frac{p_{t,k}}{z_t} = \alpha \beta E_t \left[ \frac{p_{t+1,k-1}}{z_{t+1}} \right] + (1 - \alpha) \beta E_t \left[ u' \left( \sum_{j=1}^{n} p_{t+1,j-1} b_{t,j}^i + \delta \right) p_{t+1,k-1} \right] \quad (16)$$

We restrict our attention to the symmetric equilibrium, where all employed agents hold the same quantities of bonds for maturities $k = 1, \ldots, n$. These quantities are only determined by preference parameters and aggregate variables. We hence denote by $b_{t,k}^e$ the quantity of $k$-period bonds held by employed agents at date $t$. In consequence, the consumption level $c_{t}^{eu}$ is identical across agents who are unemployed at date $t$ and who were employed at date $t - 1$. Equation (13) therefore becomes:

$$c_{t}^{eu} = \sum_{k=1}^{n} p_{t,k-1} b_{t-1,k}^e + \delta \quad (17)$$

### 3.2 Market clearing

Because all employed agents hold the same quantity of securities for each maturity, and unemployed agents do not hold any assets, bond-market equilibrium (11) is simply:

$$\omega^e b_{t,k}^e = B_k, \ \forall k = 1, \ldots, n. \quad (18)$$

where $\omega^e$ is the share of employed agents in the population, given by (1).
3.3 Pricing equations

Since all employed agents hold the same quantity of securities, we are able to derive the simple Euler pricing equations, which are the central equations of this paper. Using the Euler equation (16) and the market equilibria (18), we obtain, for all $t \geq 0$ and $k \in \{1, \ldots, n\}$:

$$
\frac{p_{t,k}}{z_t} = \alpha \beta E_t \left[ \frac{p_{t+1,k-1}}{z_{t+1}} \right] + (1 - \alpha) \beta E_t \left[ p_{t+1,k-1} u' \left( \delta + \frac{1}{\omega e} \sum_{j=1}^{n} p_{t+1,j-1}B_j \right) \right] \quad (19)
$$

The previous equations pin down the price of any bond as a function of the current and next aggregate state, all future prices, and the bond supply. Each of these equations shows the price of a $k$–period bond as the sum of two distinct terms, depending on the employment status of the agent in the next period. If the agent stays employed (which occurs with probability $\alpha$), then labor supply freely adjusts to set equal the marginal utility of consumption and the inverse of the real wage $1/z_{t+1}$. This term is the only one that would apply were markets to be complete and were agents fully able to smooth out idiosyncratic income shocks. The second term on the right-hand side of equation (19) reflects the liquidation risk, associated with the possibility that the agent could be hurt by an unfavorable change in employment status. Bond quantities directly affect prices through their effect on the value of the liquidated portfolio, which in turn feeds back into current equilibrium prices.

3.4 Conjectured asset price structure

We focus on the equilibrium where bond prices depend only on the realization of aggregate shocks. Following the literature on asset pricing with a finite state space (Mehra and Prescott (1985) amongst others), we conjecture the following expression for bond prices:

$$
\forall t \geq 0, \forall k \in \{1, \ldots, n\}, \forall s \in \{h, l\} \quad p_{t,k}^s = C_k^s z_t^s, \quad (20)
$$

where the $C_k^s$s are constant. This conjectured price structure exhibits a form of stationarity, as bond prices depend only on their maturity and the current aggregate state. As a conse-
quence, there are two yield curves, one for each value of the aggregate state. Our existence proof will thus consist in showing that such a stationary equilibrium exists.

The pricing equations (19) expressed in both states $h$ and $l$ provide the expressions for $C^h_k$ and $C^l_k$ for $k \geq 1$. We introduce the following notation:

\[
\begin{bmatrix}
C^h_0 \\
C^l_0
\end{bmatrix} \equiv \begin{bmatrix}
1/z^h \\
1/z^l
\end{bmatrix}
\]  \hspace{1cm} (21)

\[
\begin{cases}
u^h \equiv u' \left( \delta + \sum_{i=0}^{n-1} C^h_i z^h B_{i+1}/\omega e \right) \\
u^l \equiv u' \left( \delta + \sum_{i=0}^{n-1} C^l_i z^l B_{i+1}/\omega e \right)
\end{cases}
\]  \hspace{1cm} (22)

The price structure (19) can then be written compactly in a recursive form as follows, for $k = 1, \ldots, n$:

\[
\begin{bmatrix}
C^h_k \\
C^l_k
\end{bmatrix} = \beta T \cdot \begin{bmatrix}
\alpha + (1-\alpha) z^h u^h & 0 \\
0 & \alpha + (1-\alpha) z^l u^l
\end{bmatrix} \cdot \begin{bmatrix}
C^h_{k-1} \\
C^l_{k-1}
\end{bmatrix}
\]  \hspace{1cm} (23)

This system provides $2 \times n$ equations that determine the $2 \times n$ coefficients \{\(C^h_k, C^l_k\)\}_{k=1,\ldots,n}. This system is not linear, because the whole price structure appears in each coefficient $u^h$ and $u^l$.

The yield to maturity of a bond with maturity $k = 1, \ldots, n$ in state $s = h, l$ and in period $t \geq 0$ is defined by the usual expression $r_{t,k}^s = -k^{-1} \ln p_{t,k}^s$. Interest rates are supposed to be continuously compounded. The average yield curve is the sum of yield curves in states $h$ and $l$ weighted by the average frequency of aggregate state $h$ and $l$ (given by the matrix $T$). The average yield $r_{t,k}$ of maturity $k$ at date $t$ is thus:

\[
r_{t,k} = \frac{1 - \pi^l}{2 - \pi^l - \pi^h} r_{t,k}^h + \frac{1 - \pi^h}{2 - \pi^l - \pi^h} r_{t,k}^l.
\]  \hspace{1cm} (24)

This simple recursive structure is now used first to prove the existence of a stationary equilibrium, and then to analyze how the equilibrium yield curve is affected by bond supply.
4 Existence of the equilibrium

We begin with imposing some general conditions, which ensure that our equilibrium with limited agent-heterogeneity exists. We then establish the existence of an equilibrium with an arbitrarily large number of bond maturities. This is done by first deriving the equilibrium yield curve under aggregate certainty and zero net supply for all maturities. We then show that the yield curve is continuous with respect to both the introduction of small aggregate uncertainty and small and positive bond supply. This allows us to extend our existence result to a more general case.

4.1 General conditions for the equilibrium to exist

The stationary distribution with four agent types is constructed under the assumption that unemployed agents are always borrowing-constrained, while no employed agent is. We now derive the conditions under which this is indeed the case.

4.1.1 Conditions on agents’ initial wealth

In order to avoid the complications related to the transitional adjustment of agents’ wealth levels towards the steady state, we assume that, at the beginning of period 0, employed agents hold an initial quantity of bonds $b_{e_{-1,k}}^{e} = B_k/\omega^e$ with probability $\alpha$, and hold $b_{e_{-1,k}}^{ue} = 0$ with probability $1 - \alpha$. Unemployed agents hold $b_{u_{-1,k}}^{u} = 0$ with probability $\rho$, and $b_{u_{-1,k}}^{eu} = B_k/\omega^e$ with probability $1 - \rho$. As a result, from an ex ante point of view, agents are employed with positive bond holdings with probability $\alpha \omega^e$, and unemployed with positive bond holdings with probability $(1 - \rho) \omega^u$. The initial joint distribution of labor force status and bond holdings is thus identical to the stationary distribution.

4.1.2 Conditions on parameter values

Agents who are unemployed at both dates $t - 1$ and $t$ consume the amount $c_{u_{t+1}}^{u} = \delta$. If they become employed in the next period, which occurs with probability $1 - \rho$, then $\eta_{t+1}^i = 1$ (see Eq. (7)). Moreover, if they remain unemployed, which occurs with probability $\rho$, then from
Equations (6), (7), and (10) imply that condition (12), which precisely states that these agents are constrained, is equivalent to:

$$\forall k = 1, \ldots, n, \quad p_{t,k} u'(\delta) > \beta (1 - \rho) E_t \left[ \frac{p_{t+1,k-1}}{z_{t+1}} \right] + \beta \rho u'(\delta) E_t [p_{t+1,k-1} u'(\delta)]$$ (25)

Agents who were employed at date $t - 1$ and who become unemployed at date $t$ consume their home production $\delta$ as well as the liquidation value of their bond portfolio. From equation (12) again, these agents face binding borrowing constraints if and only if:

$$\forall k \quad p_{t,k} u' \left( \delta + \sum_{j=1}^{n} p_{t,j-1} B_j / \omega^e \right) > \beta (1 - \rho) E_t \left[ \frac{p_{t+1,k-1}}{z_{t+1}} \right] + \beta \rho u'(\delta) E_t [p_{t+1,k-1} u'(\delta)]$$ (26)

Note that inequality (26) implies (25). We thus only need to check that the inequality (26) holds, i.e. that agents who enter unemployment become constrained and liquidate their bond portfolios. This in turn implies that unemployed agents who were already unemployed last period, also face binding borrowing constraints.

### 4.2 Equilibrium existence with zero net supply and no aggregate shock

If assets are in zero net supply, then $\sum_{j=1}^{n} p_{t,j-1} B_j / \omega^e = 0$. With no aggregate uncertainty, $z_t = z \ \forall t$, and equation (20) becomes $p_{t,k} = C_k z$ (i.e. the $C_k$s only depend on bond maturity). Then, substituting (20) into (19) and (26) and rearranging, (26) becomes:

$$(\alpha + (1 - \alpha) u'(\delta)) u'(\delta) > (1 - \rho) + \rho u'(\delta)$$

Since $u'(\delta) > 1$ (from assumption B), this condition is satisfied when $\alpha < 1$. As long as credit constraints are binding, our equilibrium exists in an economy without aggregate risk and with zero net supply.

---

9The RHS reaches its maximum $u'(\delta)$ when $\rho = 1$, and when $\alpha < 1$ we have $(\alpha + (1 - \alpha) u'(\delta)) u'(\delta) > u'(\delta)$. 

18
4.3 Continuity of the yield curve as a function of bond supply and shocks

The following proposition summarizes the regularity property of the yield curve, which will be used extensively below. We introduce the following notation: \( B \) is the vector of bond quantities for the \( n \) maturities: \( B = [B_1 \ldots B_n]^\top \), \( Z \) the vector of wages (or equivalently productivities) \( Z = [z^l \ z^h]^\top \), and \( C \) the vector of coefficients for both states \( h \) and \( l \) and the \( n \) maturities: \( C = [C^h_1 \ C^l_1 \ldots C^h_n \ C^l_n]^\top \). \( 1_n \) and \( 0_n \) are vectors of length \( n \) containing only 1 and 0, respectively.

**Lemma 1 (Regularity of the yield curve)**

*If \( B \) is in the neighborhood of \( 0_n \) and \( Z \) in the neighborhood of \( 1_2 \), then \( C \) is a \( C^1 \) function of \( B \) and of \( Z \).*

All proofs are presented in the Appendix. Lemma 1 essentially states that, starting from a no uncertainty-zero net supply situation, a gradual increases in aggregate risk or bond supply does not cause the yield curve to jump.

4.4 Equilibrium existence in the general case

The system (23) with initial conditions (21) defines the vector of price constants \( C \) as a continuous function of bond volumes \( B \) and real wages \( Z \), when \( [B^\top \ Z^\top] \) is in a neighborhood \( V_1 \) of \( [0_n^\top \ 1_2^\top] \). Moreover, if \( [B^\top \ Z^\top] = [0_n^\top \ 1_2^\top] \), the equilibrium vector \( C \) satisfies conditions (26). By continuity, there exists a neighborhood \( V_2 \subset V_1 \) of \( [0_n^\top \ 1_2^\top] \), such that condition (26) is fulfilled if \( [B^\top \ Z^\top] \in V_2 \). We thus have the following proposition:

**Proposition 1 (Existence)**

*The equilibrium exists under the condition that both aggregate uncertainty and bond supply be small.*

If the supply of bonds of any maturities remains small, and if the variance of the aggregate shock is low enough (the process \( z \) remains around the mean 1), then the equilibrium with four agent types exists.
5 Volume effects on the yield curve

We can now state our main results regarding the impact of bond volumes on the yield curve. The following lemma characterizes the ranking of yield curves according to the aggregate state. All results are obtained for $\alpha < 1$ and $B$ small enough for the equilibrium to exist.

Lemma 2 (Ranking of yield curves)

The yield curve in the good aggregate state lies strictly below that in the bad aggregate state:

$$r^h_k < r^l_k \quad k = 1, \ldots, n.$$  

Yields in both aggregate states converge to a common limit:

$$\lim_{k \to \infty} r^l_k = \lim_{k \to \infty} r^h_k = r_{\lim}.$$  

In the good state, employed agents choose a higher level of savings than in the bad state (where their labor income is lower), leading to higher bond prices for all maturities in the good state than in the bad state.\(^{10}\)

We define the slope of the yield curve $\Delta$ as the difference between the long run yield $r_{\lim}$ (see lemma 2) and the average short yield, $r_1$ given by (24): $\Delta = r_{\lim} - r_1$. The following proposition summarizes the effect of a variation in the net supply of bonds on the level and slope of the yield curve.

Proposition 2 (Impact of bond volumes on the shape of the curve)

1) Raising the net supply of bonds of any maturity increases the level of the yield curve:

$$\frac{\partial p^s_k}{\partial B_i} < 0$$

for $i, k = 1, \ldots, n$ and $s = h, l$.

Moreover, if $\alpha$ is close to 1 then:

2) Raising the net supply of bonds of any maturity increases the slope of the yield curve:

$$\frac{\partial \Delta}{\partial B_i} > 0$$

for $i = 1, \ldots, n$.

The first statement in Proposition 2 establishes that a larger bond supply of any maturity decreases the prices of bonds of all maturities (including the price of arbitrarily long bonds).

\(^{10}\)This statement is not related to the presence of borrowing constraints per se. Donaldson, Johnsen and Mehra (1990) establish the same ranking of yield curves with complete markets.
in both aggregate states, and thus shifts the yield curve upwards. This effect stems from the specific liquidity role played by bonds in this economy. Employed agents, who earn higher labor market incomes, want to self-insure against the risk of falling into unemployment. Available bonds of any maturity serve this purpose. A small supply of bonds makes such liquidity devices highly valuable and thus yields a high price, relative to a situation where bonds are abundant. Conversely, an increase in total liquidity, through for example greater bond supply (whatever the maturity), lowers the price of all bonds. In brief, incomplete markets coupled with borrowing constraints make aggregate bond demand downward-sloping (when $\alpha = 1$ no agent is ever constrained and this effect of bond supply on equilibrium prices disappears).\(^{11}\)

The second statement in Proposition 2 bears upon the change in relative bond prices induced by changes in the total supply of bonds. It is noteworthy that $\alpha$ is close to 1.\(^{12}\) The increase in the supply of bonds raises the slope of the yield curve for two reasons. First, note that agents care about the expected liquidation value of their portfolio in the next period, which is $\sum_{j=1}^{n} p_{t+1,j-1} (B_j/\omega^e)$. Take the case where the volume of bonds with maturity $k > 1$ is raised while all other volumes are kept to zero. Then, the variance of the portfolio is $\text{var} [p_{t+1,k-1}] (B_k/\omega^e)^2$, which increases with $B_k$: The liquidation value of the portfolio becomes more volatile as bond supply increases, whatever its maturity. However, long-term bonds are more likely to be liquidated before maturity. When the volatility of the portfolio raises, the risk of holding long-term bonds increases, which therefore produces a greater risk premium than for shorter bonds. Second, when the volume of bonds increases, agents are better able to self-insure. As a result, their demand for self-insurance decreases. As bonds of various maturities are imperfect substitutes for each other, because of their probability to be sold before maturity differs, the relative demand for longer bonds decreases compared to shorter bonds. Here again, both effects of the total supply of bonds on the slope vanish,\(^{13}\)

\(^{11}\) The positive effect of bond volumes on interest rates has been underlined in a number of empirical papers (Duffee (1996), Fleming (2002) among others). Other empirical work, however, has found the opposite effect (see Amihud and Mendelson (1991), for example). Jegadeesh (2002) conjectures that the negative effect of volumes on prices may occur in markets, which lack depth. In such markets, a greater bond supply may increase trading volumes, thereby reducing trading frictions and raising prices.

\(^{12}\) The empirical evidence suggests that $\alpha$ is close to 1, the case that we focus on theoretically. For example, Engel and Gruber (2001) estimate that $\alpha = 0.97$ at a quarterly frequency.
when idiosyncratic uncertainty disappears (i.e. when $\alpha = 1$).

6 Welfare

In our model, the net bond supply is set arbitrarily. Taxes simply adjust so as to satisfy the government budget constraint at all times. The equilibrium is then indexed by the quantity of bonds available in the economy. This model is simple enough to allow us to rank equilibria according to a Pareto criterion.

Proposition 3 (Pareto ranking of equilibria)

Without aggregate shocks:

1. A greater bond supply always increases, in a Pareto sense, ex ante welfare.

2. Starting from a zero net supply of bonds, a greater bond supply always increases the welfare of $ee$ and $eu$ agents, but increases the welfare of $ue$ and $uu$ agents if and only if
\[
\beta > \left[\alpha + (1 - \alpha)u'(\delta)\right]^{-1}.
\]

If from an ex ante point of view, when agents ignore their types, greater bond supply is Pareto improving, this improvement is not homogeneous across types. First, the instantaneous utilities of agents $ee$ and $uu$ are not affected. This is neutral for $uu$ since they only earn a constant income $\delta$. For $ee$ agents, the increase in taxation (which finances greater bond supply) is fully offset by the higher value of the portfolio they sell. On the other hand, $ue$ types do not have any assets to sell (they were unemployed and thus credit constrained in the preceding period) and only endure the tax increase, which makes them work more. Finally, $eu$ agents are unemployed, but do not pay any tax: They therefore benefit from a higher total value of their bond portfolio. Instantaneously, greater bond supply is a redistribution from $ue$ (who work but do not have any securities to sell) to $eu$ (who are unemployed, but liquidate a bond portfolio). Since the marginal gain of $eu$ is larger than the marginal loss of $ue$, and since both types are equally likely, the first part of the proposition is straightforward: Ex ante, a larger bond supply is always Pareto improving.
The expected welfare comparison from the point of view of date 0, when agents know their type, is less direct. The expected utility for each type balances both today’s and tomorrow’s impacts. Note that the situation of the agent ee (resp. uu) is analogous to that of eu (resp. ue), since they are currently not affected by bond supply. On the one hand, an ue agent’s utility is positively affected by greater bond supply only if their current loss is offset by the gain of possibly becoming eu tomorrow: If the agent is patient enough, the increase in the bond supply will be welfare improving. On the other hand, the gain today of an eu agent is mitigated by the probability of becoming ue tomorrow and thereby possibly suffering from a greater bond supply. However, instantaneous utilities imply that this possible loss is not large enough to offset today’s certain gain. Finally, both eu and ee agents, always benefit from an increase in bond supply, whereas ue and so uu may suffer from it, if they are not patient enough.

7 Concluding remarks

This paper has analyzed the yield curve implications of a simple dynamic general equilibrium model with incomplete markets, where agents face both idiosyncratic and aggregate shocks. Our focus on an equilibrium where assets are fully liquidated following an adverse idiosyncratic shock allowed us to derive analytical expressions for bond prices at any maturity.

Two properties of bonds stand out in our liquidity constrained economy. First, bonds are used as liquidity means by private agents attempting to smooth out labour income fluctuations. Second, bonds of different maturities are imperfect substitutes for each other (due to varying risks of early liquidation) and thus command different risk compensation in equilibrium. The first property implies that the entire yield curve is raised following an increase in the supply of any type of bonds, while under the second it triggers a steepening of the yield curve (as long bonds become comparatively riskier when the overall variance of the portfolio increases).

Other properties of the yield curve can be derived within this framework, and are available in an extended version of this paper. First, the imperfect substituability of bonds under
market incompleteness contributes to the rejection of the ‘expectation hypothesis’. Second, the interactions of idiosyncratic and aggregate risks imply that the variance of the yield curve is affected by the total supply of bonds. Our model could also be extended in several directions in order to identify new economic determinants of the yield curve under incomplete markets. For example, a natural extension would be to introduce money and examine the effects of monetary policy on the yield curve.
A Proof of proposition 1

We prove that $C^s_k$ are $C^1$ functions of $B_i$ and $z^h, z^l$ for $s = h, l$ and $k, i = 1, \ldots, n$. We define the following matrices ($u^d$ and $u^h$ are defined in (22)):

$$C = \begin{bmatrix} C^h_n & C^l_n & \cdots & C^h_1 & C^l_1 & C^h_0 & C^l_0 \end{bmatrix}^\top$$

$$X = \begin{bmatrix} z^h & z^l & B^\top \end{bmatrix} \text{ with } B = [B_n & \cdots & B_1]^\top$$

$$M(C, X) = \beta \begin{bmatrix} \pi^h (\alpha + (1 - \alpha) z^h u^h) & (1 - \pi^h) (\alpha + (1 - \alpha) z^l u^l) \\
(1 - \pi^l) (\alpha + (1 - \alpha) z^h u^h) & \pi^l (\alpha + (1 - \alpha) z^l u^l) \end{bmatrix}$$

(27)

We write now the price structure (23) using the preceding notations (where $0_{m \times n}$ is the $m \times n$ null matrix; if $m = 0$ or $n = 0$, then the matrix has no dimension):

$$f(C, X) \equiv C - \begin{bmatrix} 0_{2 \times 2} & M(C, X) & 0_{2 \times 2} & \cdots & 0_{2 \times 2} \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & M(C, X) & 0_{2 \times 2} \\
0_{2 \times 2} & \cdots & 0_{2 \times 2} \end{bmatrix} C - \begin{bmatrix} 0 \\
\vdots \\
1/z^h \\
1/z^l \end{bmatrix} = 0_{(2n+2) \times 1}$$

Since $u'$ is $C^1$ on $\mathbb{R}$, $M$ and $f$ are also $C^1$ in $(C, X)$ on $\mathbb{R}^{2n+2} \times \mathbb{R}^{n+2}$. Before using the implicit function theorem to show that $C$ is $C^1$ in $(B, X)$, we prove that the Jacobian $Df_Y = (\frac{\partial f}{\partial C^h_n}, \frac{\partial f}{\partial C^l_n}, \ldots, \frac{\partial f}{\partial C^h_{n-i}}, \ldots, \frac{\partial f}{\partial C^l_n}, \frac{\partial f}{\partial C^l_0})$ of $f$ relative to $C$ is invertible.

As in (22) for $u^h$ and $u^l$, we define $u'^h$ and $u'^l$ as follows:

$$u'^h \equiv u'' \left( \delta + \sum_{i=0}^{n-1} C^h_i z^h B_{i+1}/\omega^c \right) \quad \text{and} \quad u'^l \equiv u'' \left( \delta + \sum_{i=0}^{n-1} C^l_i z^l B_{i+1}/\omega^c \right)$$

(28)

We now consider the partial derivatives of $f$ relative to $C^a_{n-i}$ ($i = 0, \ldots, n - 1$). First,
derivatives with respect to $C_n^h$ and $C_n^l$ are:

\[
\frac{\partial f}{\partial C_n^h} = \begin{bmatrix} 1 \\ 0 \\ 0_{2n \times 1} \end{bmatrix}, \quad \frac{\partial f}{\partial C_n^l} = \begin{bmatrix} 0 \\ 1 \\ 0_{2n \times 1} \end{bmatrix}
\]

Second, the derivatives relative to $C_n^{h-i}$ and $C_n^{l-i}$ for $1 \leq i \leq n$ are:

\[
\frac{\partial f}{\partial C_n^{h-i}} = \begin{bmatrix} 0_{2(i-1) \times 1} \\ -\beta (\alpha + (1-\alpha)z^h u^{h'}) (1 - \pi^h) \\ 1 \\ 0 \\ 0_{2(n-i) \times 1} \end{bmatrix}, \quad \frac{\partial f}{\partial C_n^{l-i}} = \begin{bmatrix} \pi^h C_{n-1}^h \\ (1-\pi^l) C_{n-1}^h \\ \vdots \\ \pi^h C_0^h \\ (1-\pi^l) C_0^h \\ 0_{2 \times 1} \end{bmatrix}
\]

The Jacobian $Df_Y$ is written as the sum of an upper triangular matrix with only 1 on its diagonal and a matrix, which is equal to 0 when $B = 0$. Close to the point of zero net supply, the Jacobian is invertible and $C$ is a $C^1$ function of $B$ and of $\{z^h, z^l\}$. QED.
B Proof of lemma 2

B.1 Ranking of yield curves.

We prove by inference in the zero net supply case \((B = 0)\) that \(C_k^h z^h > C_k^l z^l\) for \(k = 1, \ldots, n\) if \(\pi^h + \pi^l > 1\) using price definitions (23).

1. The result holds for \(k = 1\). Substituting \(C_1^h\) and \(C_1^l\) using (23), we find that \(C_1^h z^h > C_1^l z^l\) is equivalent to

\[
\alpha \left( \pi^h - (1 - \pi^l) \frac{z^l}{z^h} + \frac{z^h}{z^l} (1 - \pi^h) - \pi^l \right) > (1 - \alpha) u'(\delta) \left( \pi^l z^l + (1 - \pi^l) z^l - \pi^h z^h - (1 - \pi^h) z^h \right)
\]

(29)

Since \(z^h \geq z^l\) and \(\pi^h + \pi^l - 1 > 0\), we can check that the left-hand side is strictly positive whereas the right-hand side is strictly negative.

2. For a given maturity \(k \geq 2\), let us suppose that the result holds for the previous maturity: \(C_{k-1}^h z^h > C_{k-1}^l z^l\). Proving our result \(C_k^h z^h > C_k^l z^l\) is equivalent to:

\[
\alpha \left( \left( \pi^h - (1 - \pi^l) \frac{z^l}{z^h} \right) \frac{z^h C_{k-1}^h}{z^l C_{k-1}^l} + \frac{z^h}{z^l} (1 - \pi^h) - \pi^l \right) \geq (1 - \alpha) u'(\delta) \left( \pi^l z^l + ((1 - \pi^l) z^l - \pi^h z^h) \frac{z^h C_{k-1}^h}{z^l C_{k-1}^l} - (1 - \pi^h) z^h \right)
\]

(30)

First, consider the right-hand side RHS. Since \(z^h \geq z^l\) and \(\pi^h + \pi^l - 1 > 0\), we have \((1 - \pi^l) z^l - \pi^h z^h < 0\). By assumption \(\frac{z^h C_{k-1}^h}{z^l C_{k-1}^l} \geq 1\), the RHS can thus be written as:

\[
\text{RHS} < (1 - \alpha) u'(\delta) \left( \pi^l z^l + (1 - \pi^l) z^l - \pi^h z^h - (1 - \pi^h) z^h \right) < (1 - \alpha) u'(\delta) (z^l - z^h) < 0
\]

Second, consider the LHS. Using the same argument as for the RHS, we obtain:

\[
\text{LHS} > \alpha \left( \pi^h - (1 - \pi^l) \frac{z^l}{z^h} + \frac{z^h}{z^l} (1 - \pi^h) \right) > \alpha \left( z^h - z^l \right) \left( \frac{1 - \pi^h}{z^l} + \frac{1 - \pi^l}{z^h} \right) > 0
\]
By inference, we obtain $C_k^h z^h > C_k^l z^l$ for $k = 1, \ldots, n$ if $\pi^h + \pi^l > 1$. The result is true in zero supply and still holds, by continuity, for small bond volumes. QED.

### B.2 Value of the long-run interest rate.

We determine the common value towards which yields converge in both states. We diagonalize the matrix $M(C, X)$ defined in (27) as $M(C, X) = \beta Q D Q^{-1}$. The matrices $Q$ and $D$ are defined as:

$$Q \equiv \begin{bmatrix}
\frac{\alpha (\pi^h - \pi^l) - (\alpha - 1) (z^h u^h \pi^h - z^l u^l \pi^l) - H}{2(1 - \pi^l)} & \frac{\alpha (\pi^h - \pi^l) - (\alpha - 1) (z^h u^h \pi^h - z^l u^l \pi^l) + H}{2(1 - \pi^l)} \\
1 & 1
\end{bmatrix}$$

$$D \equiv \frac{1}{2} \begin{bmatrix}
\alpha (\pi^h + \pi^l) + (1 - \alpha) (z^h u^h \pi^h + z^l u^l \pi^l) - H & 0 \\
0 & \alpha (\pi^h + \pi^l) + (1 - \alpha) (z^h u^h \pi^h + z^l u^l \pi^l) + H
\end{bmatrix}$$

$$H \equiv [(\alpha (\pi^h + \pi^l) + (1 - \alpha) (z^h u^h \pi^h + z^l \pi^l u^l))^2 - 4(z^h u^h (1 - \alpha) + \alpha) (z^l u^l (1 - \alpha) + \alpha) (\pi^h + \pi^l - 1)]^{1/2}$$

(We can check that the term under the square is always positive in the neighborhood of zero net supply). We can now diagonalize the matrix defining $C^h$ and $C^l$ and modify the system (23) and iterate it to obtain:

$$\begin{bmatrix}
C_k^h \\
C_k^l
\end{bmatrix} = \beta^k Q.D^k.Q^{-1} \begin{bmatrix}
C_0^h \\
C_0^l
\end{bmatrix} \Rightarrow \begin{bmatrix}
P_k^h \\
P_k^l
\end{bmatrix} = \beta^k \begin{bmatrix}
z^h & 0 \\
0 & z^l
\end{bmatrix} Q.D^k.Q^{-1} \begin{bmatrix}
C_0^h \\
C_0^l
\end{bmatrix}$$

Developing the preceding equality allows us to obtain an analytical expression for $P_k^h$. Noting that $H > 0$, we simplify the expression with $$\left(\frac{\alpha (\pi^h + \pi^l) - (\alpha - 1) (z^h u^h \pi^h + z^l u^l \pi^l) - H}{\alpha (\pi^h + \pi^l) - (\alpha - 1) (z^h u^h \pi^h + z^l u^l \pi^l) + H}\right)^k \to 0 \text{ as } k \to \infty.$$ As a consequence, the price $P_k^h$ satisfies:

$$\lim_{k \to \infty} \log \left[ P_k^h \left(\frac{1}{H^{2^k - 1}} \beta^k \right)^{-1} \left(\alpha (\pi^h + \pi^l) - (\alpha - 1) (z^h u^h \pi^h + z^l u^l \pi^l) + H\right)^{-k} \right]$$

$$= \log \left[ H - \left(\alpha (\pi^l - \pi^h) + (\alpha - 1) (z^h u^h \pi^h - z^l u^l \pi^l)\right) + 2(\alpha + z^l u^l (1 - \alpha))(1 - \pi^h) \frac{z^h u^h}{z^l u^l} \right]$$
As \( r^h_k = -\frac{1}{k} \log P^h_k \), we finally obtain the expression for \( r^h_k \) when its maturity goes to infinity. By a simple symmetry argument, we obtain the same expression for the common limit:

\[
\lim_{k \to \infty} r^h_k = \lim_{k \to \infty} r^l_k = \tilde{r}^\text{lim} = -\log \beta - \log \frac{\alpha (\pi^h + \pi^l) + (1 - \alpha) (z^h u^h \pi^h + z^l u^l \pi^l) + H}{2}
\]

(31)

\[\text{C} \quad \text{Proof of proposition 2}\]

\[\text{C.1 Impact of bond volumes on prices.}\]

We prove by inference that bond volumes reduce prices and increase yields. We prove the result for \( C^h_k \); the method is the same for \( C^l_k \). We begin by expressing the derivative of \( C^h_k \) relative to \( B_i \) for \( 1 \leq k, i \leq n \) (\( u''_h \) and \( u''_l \) are defined in (28)):

\[
\frac{\partial C^h_k}{\partial B_i} = \alpha \beta \left( \pi^h \frac{\partial C^h_{k-1}}{\partial B_i} + (1 - \pi^h) \frac{\partial C^l_{k-1}}{\partial B_i} \right) + (1 - \alpha) \beta \left( \pi^h \frac{\partial C^h_{k-1}}{\partial B_i} z^h u^h + (1 - \pi^h) \frac{\partial C^l_{k-1}}{\partial B_i} z^l u^l \right) + (1 - \alpha) \beta \pi^h C^h_{k-1} (z^h)^2 \left( \sum_{j=1}^{n} \frac{\partial C^h_{j-1}}{\partial B_i} B_j + C^h_{i-1} \right) u^m^h
\]

\[
+ (1 - \alpha) \beta (1 - \pi^h) C^l_{k-1} (z^l)^2 \left( \sum_{j=1}^{n} \frac{\partial C^l_{j-1}}{\partial B_i} B_j + C^l_{i-1} \right) u^m^l \]

(32)

1. The result holds for \( k = 1 \), since the equation (32) yields the following first-order approximation for small levels of bond supply:

\[
\frac{\partial C^h_1}{\partial B_i} \approx (1 - \alpha) \beta \left[ \pi^h C^h_{k-1} (z^h)^2 C^h_{i-1} u^m^h + (1 - \pi^h) C^l_{k-1} (z^l)^2 C^l_{i-1} u^m^l \right]
\]

2. We suppose that the result holds for \( k - 1 \) and \( \frac{\partial C^h_{k-1}}{\partial B_i}, \frac{\partial C^l_{k-1}}{\partial B_i} < 0 \). Since \( C^s_j \) is a \( C^1 \) function of \( B_i \), \( \frac{\partial C^s_j}{\partial B_i} \) is continuous in \( B_i \) and \( B_j \frac{\partial C^s_j}{\partial B_i} (s = h, l) \) is negligible relative to \( C^s_i \) for small levels of bond supply. Eq. (32) implies that \( \frac{\partial C^h_i}{\partial B_i} < 0 \). Greater bond supply decreases prices. \( Q.E.D. \)
C.2 Impact of bond volumes on the slope.

We prove that larger bond volumes steepen the curve. Using the expression (31) for $\bar{r}^{\lim}$ and the one for $r_1$ (the price structure (23) and (24)), the derivative of the slope relative to $B_j$ for $j \leq n$ is:

$$\frac{\partial}{\partial B_j} \Delta = (1 - \alpha) \frac{(1 - \pi^h)(1 - \pi^l)(z^l - z^h)}{2 - \pi^h - \pi^l} \times \left( \frac{z^l}{\pi^l z^h + (1 - \pi^l) z^l} \frac{\partial u^l}{\partial B_j} - \frac{z^h}{(1 - \pi^h) z^h + \pi^h z^l} \frac{\partial u^h}{\partial B_j} \right) + O \left( (1 - \alpha)^2 \right)$$

with (we also provide expressions when bond supply is close to 0):

$$\frac{\partial}{\partial B_j} u^h = z^h \left( \sum_{k=1}^{n} \frac{\partial C^h_k}{\partial B_j} B_{k+1} + C^h_{j-1} \right) u'' \left( \delta + B_1 + z^h \sum_{k=1}^{n} C^h_k B_{k+1} \right) \approx z^h C^h_{j-1} u''(\delta)$$

$$\frac{\partial}{\partial B_j} u^l = z^l \left( \sum_{k=1}^{n} \frac{\partial C^l_k}{\partial B_j} B_{k+1} + C^l_{j-1} \right) u'' \left( \delta + B_1 + z^l \sum_{k=1}^{n} C^l_k B_{k+1} \right) \approx z^l C^l_{j-1} u''(\delta)$$

The sign of $\frac{\partial}{\partial B_j} \Delta$ depends on the sign of $A \equiv \frac{z^l}{\pi^l z^h + (1 - \pi^l) z^l} \frac{\partial u^l}{\partial B_j} - \frac{z^h}{(1 - \pi^h) z^h + \pi^h z^l} \frac{\partial u^h}{\partial B_j}$. Substituting for the derivatives allows us to write $A$ when bond supply is close to 0 as:

$$A = \left( \frac{z^l}{\pi^l (z^h - z^l) + z^l C^l_{j-1} z^l} - \frac{z^h}{\pi^h (z^l - z^h) + z^h C^h_{j-1} z^h} \right) u''(\delta)$$

As $z^h > z^l$, a sufficient condition for $A$ to be positive is $C^h_{j-1} z^h > C^l_{j-1} z^l$, which is always true. Thus, bond volumes increase the slope of the curve when $\pi^h + \pi^l > 1$. QED.

D Proof of proposition 3

We prove that: (i) Ex ante welfare increases with bond supply; and (ii) at date 0, the welfare of eu and uu agents only increases if the discount factor $\beta$ is large enough. We suppose that there is no aggregate shock and $z_t = 1$. We denote by $U$ the vector of instantaneous utility.
Using the budget constraints, it is written as:

\[
U = \begin{bmatrix}
    u(c^{ee}) - l^{ee} \\
    u(c^{ue}) - l^{ue} \\
    u(c^{eu}) - l^{eu} \\
    u(c^{uu}) - l^{uu}
\end{bmatrix} = \begin{bmatrix}
    u(u^{t-1}(1)) - 1 \\
    u(u^{t-1}(1)) - 1 - \sum_{k=1}^{n} C_{k-1} B_k \\
    u(\delta + \sum_{k=1}^{n} C_{k-1} B_k) \\
    u(\delta)
\end{bmatrix}
\]

The transition matrix \( \Omega \) for the four states \{ee, ue, eu, uu\} of the economy is as follows:

\[
\Omega = \begin{bmatrix}
    \alpha & 0 & 1 - \alpha & 0 \\
    \alpha & 0 & 1 - \alpha & 0 \\
    0 & 1 - \rho & 0 & \rho \\
    0 & 1 - \rho & 0 & \rho
\end{bmatrix} = QDQ^{-1}, \text{ with } Q = \begin{bmatrix}
    1 & 1 - \alpha & 0 & 1 - \alpha \\
    1 & 0 & \rho & 1 - \alpha \\
    1 & -\alpha & 0 & -(1 - \rho) \\
    1 & 0 & -(1 - \rho) & -(1 - \rho)
\end{bmatrix}
\]

and \( D = \text{Diag}(1 \ 0 \ 0 \ \alpha + \rho - 1) \).

The vector \( U \) of the four intertemporal utilities is

\[
U = \sum_{k=0}^{\infty} \beta^k \Omega^k U = \sum_{k=0}^{\infty} \beta^k QD^k Q^{-1} U
\]

The impact of a bond increase \( B_k \) on this intertemporal utility \( U \) is:

\[
\frac{\partial U}{\partial B_k} = C_{k-1} Q \begin{bmatrix}
    \frac{1}{1 - \beta} & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & \frac{1}{1 - \beta(\alpha + \rho - 1)}
\end{bmatrix} Q^{-1} \begin{bmatrix}
    0 \\
    -1 \\
    u'(\delta + \sum_{k=1}^{n} C_{k-1} B_k) \\
    0
\end{bmatrix}
\]

The first part of the proposition stems directly from the impact of the bond supply on instantaneous utility. Whereas in states ee and uu, utility does not change, it falls from -1 in ue and goes up from \( u' = u'(\delta + \sum_{k=1}^{n} C_{k-1} B_k) \) in eu. Since our equilibrium exists only if \( u' = u'(\delta + \sum_{k=1}^{n} C_{k-1} B_k) > 1 \), and the eu and ue states are equally probable, ex ante welfare always rises with bond supply.
To obtain the second part of the result, we expand the preceding expression of $\frac{\partial U}{\partial B_k}$:

$$\frac{\partial U}{\partial B_k} = \frac{C_{k-1}}{(2 - \alpha - \rho)(1 - \beta)(1 - \beta(\alpha + \rho - 1))} \begin{bmatrix} 
\beta(1 - \alpha)(u' - \beta(1 + \rho(u' - 1))) \\
(1 - \beta \rho)(\beta(\alpha + (1 - \alpha)u') - 1) \\
(1 - \beta \alpha)(u' - \beta(1 + \rho(u' - 1))) \\
\beta(1 - \rho)(\beta(\alpha + (1 - \alpha)u') - 1) 
\end{bmatrix}$$

Defining $\beta_0 = [\alpha + (1 - \alpha)u'(\delta + \sum_{k=1}^n C_{k-1} B_k)]^{-1}$, it is straightforward to prove that if $\beta > \beta_0$, the expected welfare of all types of agents increases with $B_k$, and if $\beta < \beta_0$, that the welfare of $ue$ and $uu$ agents falls.
References


